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Lattice and Continuum Wavelets and the Block Renormalization Group

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We obtain a resolution of the identity operator, for functions on a lattice εZ^d, which is derived from the block renormalization group. We use eigenfunctions of the terms of the decomposition to form a basis for $l_2(\varepsilon Z^d)$ and show how the basis is generated from lattice wavelets. The lattice spacing ε is taken to zero and continuum wavelets are obtained.

KEY WORDS: Renormalization group; wavelets; multiscale representation.

Recently wavelets have played an important role in mathematical physics and engineering.[1–6] Here we show how lattice wavelets arise in the context of the block field renormalization group (BFRG) as used in the analysis of lattice models in statistical mechanics and lattice regularized continuum field theory.[2–4] After showing the relation between lattice wavelets and the BFRG we take the lattice spacing to zero and obtain continuum wavelets. In the continuum, wavelets are a finite set of functions such that translations and dilations (scalings) generate a basis for functions defined on R^d. The functions on different scales are orthogonal.

In typical applications of the BFRG one has a perturbed Gaussian integral and a multiscale analysis is made by first decomposing the covariance into scales. We now give this decomposition. To fix the setting and for simplicity we consider the lattice εZ^d. We could also use a finite lattice. We define averaging operators ${}^{L^{k+1}\varepsilon}Q^{L^k\varepsilon}$ which take the average of a function in $l_2(L^k\varepsilon Z^d)$ over the L^d points of a block centered at points in $L^{k+1}\varepsilon Z$, i.e.,

$$Qf(y)=L^{-d}\sum_{|x_\alpha|<L^{k+1}\varepsilon/2} f(y+x);\quad y\in L^{k+1}\varepsilon Z^d$$

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where we suppress the superscripts. We use the Riemann sum approximates to continuum integrals in the definition of inner products and $*$ for adjoint. Denote by Q_j the composition of j averaging operators irrespective of the domain lattice and set $Q_0 = I$. Now $Q_j^* f$, $f \in l_2(L^k \varepsilon Z^d)$, has the property that it is constant on $L^{k+j}\varepsilon Z^d$ blocks in $L^k \varepsilon Z^d$ or we say it is constant on L^j blocks.

If Δ^{-1} is the covariance of the Gaussian (Δ may be minus the Laplacian or some power or any nonnegative symmetric operator), then the decomposition induced by the BFRG is

$$\Delta^{-1} = \sum_{J=0}^{n-1} [\Delta^{-1} Q_j^* \Delta_j Q_j \Delta^{-1} - \Delta^{-1} Q_{j+1}^* \Delta_{j+1} Q_{j+1} \Delta^{-1}] + \Delta^{-1} Q_n^* \Delta_n Q_n \Delta^{-1} \tag{1}$$

where $\Delta_j = (Q_j \Delta^{-1} Q_j^*)^{-1}$ is assumed to exist and is called the jth effective action. The decomposition is trivial in the sense that it is telescopic, but the form of each term is not and arises from the use of the RG. Also the decay and smoothness of the kernels of the terms depend on the operator Δ.[7,8] For a decomposition of the nonsymmetric inverse Dirac operator see ref. 9.

We give a quick derivation of (1) which is based on the factorization of the generating function of a Gaussian measure. First we introduce the BFRG. If $\rho \in l_1(L^k \varepsilon Z^d)$ then we define the RG transformation $T: l_1(L^k \varepsilon Z^d) \to l_1(L^{k+1} \varepsilon Z^d)$ by

$$T\rho(\psi) = \int \delta(\psi - Q\phi)\, \rho(\phi)\, D\phi \tag{2}$$

where

$$\delta(\psi - Q\phi) = \prod_{y \in L^{K+1}\varepsilon Z^d} \delta(\psi(y) - Q\phi(y)), \quad D\phi = \prod_{x \in L^k \varepsilon Z^d} d\phi(x)$$

and we suppress domain indices. We let T^l be a composition of l transformations. The transformation T satisfies the important normalization property

$$\int T\rho(\psi)\, D\psi = \int \rho(\phi)\, D\phi \tag{3}$$

It is convenient to write (1) as

$$\Delta^{-1} = \sum_{j=0}^{n-1} M_j \Gamma_j M_j^* + M_n \Delta_n^{-1} M_n^* \tag{4}$$

where $M_j = \Delta^{-1}Q_j^*\Delta_j$ and $\Gamma_j = \Delta_j^{-1} - \Delta_j^{-1}Q^*\Delta_{j+1}Q\Delta_{j+1}Q\Delta_j^{-1}$. We call M_j the jth minimizer; $\phi = M_j\psi$ minimizes the quadratic form $1/2(\phi, \Delta\phi)$ subject to the constraint $Q_j\phi = \psi$. We call Γ_j the jth fluctuation covariance and $Q\Gamma_j = 0$. We show by induction the factorization of the generating function

$$\exp[\tfrac{1}{2}(J, \Delta^{-1}J)]$$
$$= \exp\left[\tfrac{1}{2}\left(J, \sum_{j=0}^{n-1} M_j\Gamma_j M_j^* J\right)\right]\exp[\tfrac{1}{2}(J, M_n\Delta_n^{-1}M_n^*J)]$$

which implies (1). Using (3), we have

$$\exp[\tfrac{1}{2}(J, M_n\Delta_n^{-1}M_n^*J)]$$
$$= \int \exp(J, M_n\phi)\exp[-\tfrac{1}{2}(\phi, \Delta_n\phi)\, D\phi/(J=0)]$$
$$= \int D\psi\, \delta(\psi - Q\phi)\left(\int \{\exp(J, M_n\phi)\exp[-\tfrac{1}{2}(\phi, \Delta_n\phi)]\}\, D\phi/(J=0)\right) \tag{5}$$

Make the change of variables $\phi = m_{n+1}\psi + \eta \to \eta$, where $m_{n+1} = \Delta_n^{-1}Q^*\Delta_{n+1}$, to obtain, noting that $M_n m_{n+1} = M_{n+1}$, $Qm_{n+1} = I$, and $\Delta_n = M_n^*\Delta M_n$, and that, for $C\eta = 0$, $(\eta, \Delta_n m_{n+1}\psi) = (C\eta, \Delta_{n+1}\psi) = 0$,

$$\exp[\tfrac{1}{2}(J, M_n\Delta_n^{-1}M_nJ)]$$
$$= \int \{\exp[(J, M_{n+1}\psi) - \tfrac{1}{2}(\psi, \Delta_{n+1}\psi)]\}\, D\psi/(J=0))$$
$$\times\left[\int \{\exp(J, M_n\eta)\exp[-\tfrac{1}{2}(\eta, \Delta_n\eta)]\}\, \delta(Q\eta)\, D\eta/(J=0)\right] \tag{6}$$

which establishes (5) for $n+1$. Note that a Lagrange multiplier calculation of the integral in

$$\exp[\tfrac{1}{2}(K, \Gamma_j K)] = \int \{\exp(K, \eta)\exp[-\tfrac{1}{2}(\eta, \Delta_j\eta)]\}\, \delta(Q\eta)\, D\eta/(K=0) \tag{7}$$

gives the formula for Γ_j. For later use we give the momentum representation for Δ_j and M_j.

For

$$M_k = \Delta^{-1}Q_k^*\Delta_k\colon\quad l_2(L^k\varepsilon Z^d) \to l_2(\varepsilon Z^d)$$

we write

$$f(z)=\sum_{x} M_k(z_1 x)\, g(x)(L^k\varepsilon)^d$$

so that

$$M_k(z, x)=(2\pi)^{-d} \sum_{\substack{l'=2\pi m\\ |m|<L^k/2}} \int_{|p'_\alpha|<\pi/L^k\varepsilon} e^{i(p'+l')z-ip'x} R(p', l')\, dp', \qquad z\in \varepsilon Z^d$$

where $m \in Z^d$,

$$R(p', l')=\tilde{\Delta}^{(0),\varepsilon-1}(p'+l') \prod_{\mu=1}^{d} \frac{\sin((p'_\mu+l'_\mu)/2)\, L^k\varepsilon}{L^k \sin((p'_\mu+l'_\mu)/2)\,\varepsilon} \tilde{\Delta}^{(k),L^k\varepsilon}(p')$$

and for

$$\Delta_k^{-1}=Q_k\Delta^{-1}Q_k^*: \quad l^2(L^k\varepsilon Z^d)\to l^2(L^k\varepsilon Z^d)$$

we have

$$h(x)=\sum_{x'} \Delta_k^{-1}(x, x')\, r(x')\, L^k\varepsilon$$

$$\tilde{\Delta}^{(k),L^k\varepsilon^{-1}}(p')=\sum_{l'} \left[\prod_{\mu=1}^{d} \frac{\sin((p'_\mu+l'_\mu)/2)\, L^k\varepsilon}{L^k \sin((p'_\mu+l'_\mu)/2)}\right]^2 \tilde{\Delta}^{(0),\varepsilon-1}(p'+l')$$

where

$$\Delta_k^{-1}(x, x')=(2\pi)^{-d} \int_{|p'_\alpha|<\pi/L^k\varepsilon} \tilde{\Delta}^{(k),L^k\varepsilon-1}(p')\, e^{ip'(x-x')}\, dp', \quad x, x'\in L^k\varepsilon Z^d$$

and

$$\Delta^{-1}(z, z')=(2\pi)^{-d} \int_{|p_\alpha|<\pi/\varepsilon} \tilde{\Delta}^{(0),\varepsilon-1}(p)\, e^{ip(z-z')}\, dp$$

From the decomposition (1) we obtain a decomposition of the identity by multiplying on the left and right by $\Delta^{1/2}$, i.e.,

$$I=\sum_{j=0}^{n-1} [\Delta^{-1/2}Q_j^*\Delta_j Q_j\Delta^{-1/2}-\Delta^{-1/2}Q_{j+1}^*\Delta_{j+1}Q_{j+1}\Delta^{-1/2}]$$
$$+\Delta^{-1/2}Q_n^*\Delta_n Q_n\Delta^{-1/2}\equiv \sum_{j=0}^{n-1} P_j+R_n \tag{8}$$

Now an easy calculation shows that P_j and R_n are mutually commuting orthogonal projections, i.e.,

$$\begin{gathered} P_j^2 = P_j, \qquad P_j^* = P_j, \qquad R_n^2 = R_n, \qquad R_n^* = R_n \\ P_j P_k = P_k P_j, \qquad P_j R_n = R_n P_j \end{gathered} \tag{9}$$

Thus, a function is decomposed into scales by writing

$$f = \sum_{j=0}^{n-1} P_j f + R_n f \tag{10}$$

with all terms mutually orthogonal. To see the connection with wavelets, we find the eigenfunctions of the P_j and R_n. A short calculation shows that

$$f_j = \Delta^{-1/2} Q_j^* \Delta_j u = \Delta^{1/2} M_j u, \qquad Qu = 0 \tag{11}$$

is an eigenfunction of P_j and

$$h_n = \Delta^{-1/2} Q_n^* v = \Delta^{1/2} M_n \Delta_n^{-1} v \tag{12}$$

is an eigenfunction of R_n. From (11) and (12) we see that $\Delta^{1/2} f_j$ is constant on $L^j\varepsilon$ blocks and $\Delta^{-1/2} f_j$ has average zero on $L^k\varepsilon$ blocks for $k > j$. Also, $\Delta^{1/2} h_n$ is constant on $L^n\varepsilon$ blocks. Furthermore, if Δ is taken as a positive power of the negative Laplacian, the smoothness of the eigenfunction f_j increases with the power, since f_j is in the range of $\Delta^{-1/2}$. The special case $\Delta = I$ results in Haar basis functions.

Let us consider translations of f_j. If $Qu = 0$ and u_a is the translate of u by a, then $f_j^{(a)} = \Delta^{-1/2} Q_j \Delta_j u_a$ is also an eigenfunction of P_j, but not necessarily orthogonal to f_j. Now we show that $f_j^{(a)}$ is in fact the translate by a of f_j. Letting T_a be the translation operator by a, we find

$$T_a Q_j = Q_j T_a \tag{13}$$

for a belonging to the range lattice of Q_j, so that

$$f_j^{(a)} = T_a f_j$$

since T_a commutes with Δ and Δ_j as seen by (13) and its adjoint. Thus, within each scale we generate eigenfunctions by the translation of eigenfunctions. We calculate

$$(f_j^a, f_j^b) = (u_a, \Delta_j u_b), \qquad (h_n^a, h_n^b) = (v_a, \Delta_{n+1}^{-1} v_b)$$

and since Δ_j and Δ_{n+1}^{-1} are positive, we can form an orthogonal set in each scale, for example, the jth by $e_\alpha = S_{\alpha\beta}^{-1/2} f_j^{(\beta)}$, where S is the matrix with matrix elements $S_{ab} = (u_a, \Delta_j u_b)$.

What we have done up to now is to decompose a function into different scales and in each scale we can take the eigenfunctions to be translates of each other. What can we say about the relation between eigenfunctions on different scales? In contrast to the continuum, the dilation of an eigenfunction to a new scale is not an eigenfunction on the new scale. This is due to the fact that in momentum space the operators $\tilde{\Delta}^{-1/2}$, $\tilde{\Delta}$ (the tilde denotes the Fourier transform) are not homogeneous functions as are their continuum counterparts. However, we can generate a basis for $\sum_{j=0}^{n-1} P_j l_2(\varepsilon Z^d)$ as follows:

1. On εZ^d take $L^d - 1$ functions u_α supported on the L block at zero such that $Qu_\alpha = 0$. The f_0's associated with these u_α by (11) and their translates by multiples of $L\varepsilon$ form a basis for $P_0 l_2(\varepsilon Z^d)$,

2. For the jth scale dilate the above set u_α by L^j, i.e., $u_\alpha^j(y) = u_\alpha(y/L^j)$, $y \in L^j \in Z^d$, then form the f_j with the u_α^j by (11). The translates of these f_j by multiples of $L^j\varepsilon$ then form a basis for $P_j l_2(\varepsilon Z^d)$.

The set of functions described above are our lattice wavelets.

For dimension $d = 1$, $L = 2$ these wavelets are the Lemarie functions.[10] However, for $d > 1$ the construction provides wavelets that are not tensor products of the one-dimensional wavelets. For explicit momentum representations for the kernels of the operators Δ_k, M_k, and Γ_k see ref. 7.

The above decomposition is typical for applications to lattice regularized quantum field theory models. In these models the RGT is applied until the unit scale is reached. We now relate these results to applications of the BFRG to unit lattice models of statistical mechanics,[2,7,10–12] where one is interested in the long-range behavior of correlation functions. In this case an averaging operator, also denoted by Q, is defined on functions on Z^d into itself, by

$$Qf(y) = L^\alpha L^{-d} \sum_{|x_\alpha| < L/2} f(Ly + x), \qquad y \in Z^d$$

The parameter α is taken such that the average field has the same long-range behavior as the original field. For example, if Δ is the negative of the Laplacian, $L^\alpha = L^{(d-2)/2}$. The RGT is obtained similary to Eq. (2) but for densities on Z^d. The previously obtained properties of the decomposition of Δ^{-1}, the identity, and the wavelets continue to hold (see also ref. 10 for the construction of wavelets in this context). Furthermore, it is shown in ref. 7 that in the decomposition of $\Delta^{-1}(x, x')$ in Eq. (1) the jth term decays like $L^{-j(d-2)} \exp(-L^{-j}|x - x'|)$, so it is roughly the contribution of the momentum scale L^{-j}. Here Δ_n converges to a Gaussian fixed point Δ_∞ of the RGT.

For models which are perturbations of Gaussians, multiscale formulas are found in refs. 11 and 12 for the generating and correlation functions after the application of an RGT. Furthermore, the "orthogonality between scales" or wavelet structure implicit in the decomposition of Eq. (8) plays an important role in the simplicity of the formulas and allows for the control of the correlation functions for some asymptotic free models. Explicitly at the operator level the properties $M_j \Gamma_j M_j^* M_k \Gamma_k M_k^* = \delta_{jk} M_k \Gamma_k M_k^*$ and $M_n^* \Delta M_j \Gamma_j M_j^* = 0$, $j = 0, 1, \ldots, n-1$, are used. These properties follow from $P_j P_k = \delta_{jk} P_j$ and $P_j R_n = 0$, $j = 0, 1, \ldots, n-1$. In these models the density converges to the fixed point $\exp[-Z^{-1}(\phi, \Delta_\infty \phi)/2]$ and the long-range behavior of the two-point function is $Z\Delta^{-1}$. We point out that other commonly used decompositions of Δ^{-1} do not have the "orthogonality between scales" property.

We now pass to the continuum in Eqs. (1) and (8) by taking the number of steps $n \to \infty$ and the lattice spacing $\varepsilon \to 0$ in such a way that $L^n \varepsilon = 1$. Writing the Q_j, $j = n + s$, as

$$Q_{n+s} f(y) = \frac{1}{L^{(n+s)d} \varepsilon^d} \sum_{|x_\alpha| < L^{(n+s)}\varepsilon/2} \varepsilon^d f(y+x), \quad y \in L^{n+s} \varepsilon Z^d$$

and taking the continuum limit, we define the averaging operator $C_s \colon L^2(R^d) \to l_2(L^s Z^d)$ by

$$C_s f(y) = \frac{1}{L^{sd}} \int_{|x_\alpha| < L^s/2} f(y+x)\, dx, \qquad y \in L^s Z^d$$

so that $C_s^* g(y+x) = g(y)$ for $|x_\alpha| < L^s/2$, $g \in l_2(L^s Z^d)$, and the limit of (1) becomes

$$\Delta^{-1} = \sum_{m=-\infty}^{-1} (\Delta^{-1} C_m \Delta_m C_m \Delta^{-1} - \Delta^{-1} C_{m+1}^* \Delta_{m+1} C_{m+1} \Delta^{-1}) + \Delta^{-1} C_0^* \Delta_0 C_0 \Delta^{-1} \tag{14}$$

where in (14) Δ^{-1} is the continuum operator and $\Delta_m = (C_m \Delta^{-1} C_m^*)^{-1}$ is a lattice operator on $l_2(L^m Z^d)$. We can continue telescoping the last term of (14) to go to the LZ^d, $L^2 Z^d$,... lattices, obtaining

$$\Delta^{-1} = \sum_{m=-\infty}^{\infty} (\Delta^{-1} C_m^* \Delta_m C_m \Delta^{-1} - \Delta^{-1} C_{m+1}^* \Delta_{m+1} C_{m+1} \Delta^{-1}) \tag{15}$$

and a decompositon of the identity as

$$I = \sum_{n=-\infty}^{\infty} (\Delta^{-1/2} C_n^* \Delta_n C_n \Delta^{-1/2} - \Delta^{-1/2} C_{n+1}^* \Delta_{n+1} C_{n+1} \Delta^{-1/2})$$
$$= \sum_{n=-\infty}^{\infty} P_n$$

where the P_n are mutually commuting orthogonal projections.

Letting T_a^c be the translation operator by a in the continuum, we have

$$T_a C_m = C_m^* T_a^c, \qquad a \in L^m Z^d \tag{16}$$

and, as before, letting $M_m^c = \Delta^{-1} C_m^* \Delta_m$,

$$f_m = \Delta^{-1/2} C_m^* \Delta_m u = \Delta^{1/2} M_m^c u, \qquad Ru = 0, \qquad u \in l_2(L^m Z^d) \tag{17}$$

is an eigenfunction of P_m, where $R: l_2(L^m Z^d) \to l_2(L^{m+1} Z^d)$ is the lattice averaging operator over L blocks and satisfies $C_{m+1} = RC_m$. Using (16), we obtain that

$$f_m^{(a)} = T_a^c f_m, \qquad a \in L^{m+1} Z^d$$

is an eigenfunction of P_m. Furthermore,

$$(f_m^{(a)}, f_m^{(b)}) = (u_a, \Delta_m, u_b)$$

Now we consider dilations. Let D_k^c be the continuum dilation operator defined by

$$D_k^c h(x) = h(x/L^k)$$

and D_k the lattice dilation operator; then, by Fourier transform considerations as in ref. 8,

$$D_k^c f_m = c(k, L)\, \Delta^{-1/2} C_{k+m}^* \Delta_{k+m} D_k u$$

$c(k, L)$ a constant; and thus $P_{k+m} D_k^c f_m = D_k^c f_m$, since $D_k^c f_m$ has the form (17) and $RD_k u = 0$. In words, dilating eigenfunctions gives eigenfunctions on the dilated scale.

A basis for $L^2(R^d)$ is formed as follows:

1. Take the f_0's associated by (17) with $L^d - 1$ u_α's $\in l_2(Z^d)$, $Ru_\alpha = 0$, supported on the L block at zero. The translation by multiples of L generates $P_0 L^2(R^d)$.

2. Dilating the f_0's by L^k and translating by multiples of L^k generates $P_k L^2(R^d)$.

These are the continuum wavelets.

In ref. 10 these wavelets are related to the Gaussian fixed point in the perturbation of unit lattice Gaussian models of statistical mechanics mentioned previously.

For example, if the initial action is taken as the negative Laplacian, the fixed point is

$$\tfrac{1}{2}(\phi, \Delta_{\infty}\phi) = \lim_{k\to\infty} \tfrac{1}{2}(A_k\phi, \Delta A_k\phi) = \lim_{k\to\infty} \tfrac{1}{2}(\partial^{L^{-k}}\mathscr{A}_k\phi, \partial^{L^{-k}}\mathscr{A}_k\phi)_{\eta}$$

where η is the inner product in $l_2(L^{-k}Z^d)$, A_k is the unit lattice minimizer, and

$$\mathscr{A}_k(z, x) = L^{k(d-2)/2}A_k(L^k z, x), \qquad z \in L^{-k}Z^d$$

is the canonically scaled continuum-like minimizer of ref. 7. From Appendix 1 of ref. 7 we see that $\mathscr{A}_{\infty}(z, x) = M_0^c(z, x)$, $z \in R^d$, and $\tilde{\Delta}^{\infty}(p') = \tilde{\Delta}_{\infty}(p')$ in the formula for the continuum minimizer.

The RG transformation (2) considered up to now is the $a \to \infty$ limit of the exponential RG transformation, denoted T_a, defined by replacing the δ function by $N^{-1}e^{-a|\psi - Q\phi|^2}$, where N is a normalization constant such that (3) holds.[8,9] T_a also induces a telescopic decomposition, but the terms are more complicated than those of (1) and (8) and the terms in the decomposition of I are not projections. However, as shown in ref. 8, the decay and smoothness properties of the kernels of the terms still hold.

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